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Chapter

Singularly Perturbed Parabolic Problems

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Abstract

The aim of this work is to construct regularized asymptotic of the solution of a singularly perturbed parabolic problems. Namely, in the first paragraph, we consider the case when the scalar equation contains a free term consisting of a finite sum of the rapidly oscillating functions. In the first paragraph, it is shown that the asymptotic solution of the problem contains parabolic, power, rapidly oscillating, and angular boundary layer functions. Angular boundary layer functions have two components: the first one is described by the product of a parabolic boundary layer function and a boundary layer function, which has a rapidly oscillating change. The second section is devoted to a two-dimensional equation of parabolic type. Asymptotic of the scalar equation contains a rapidly oscillating power, parabolic boundary layer functions, and their product; then, the multidimensional equation additionally contains a multidimensional composite layer function.

Keywords: singularly perturbed parabolic problem, asymptotic, stationary phase, power boundary layer, parabolic boundary layer, angular boundary layer

1. Asymptotics of the solution of the parabolic problem with a stationary phase and an additive-free member

1.1 Introduction

Singularly perturbed problems with rapidly oscillating free terms were studied in [1–3]. Ordinary differential equations with a rapidly oscillating free term whose phase does not have stationary points are studied in [1]. Using the regularization method for singularly perturbed problems [4], differential equations of parabolic type with a small parameter were studied in [2, 3] when fast-oscillating functions as a free member. The asymptotic solutions constructed in [1–3] contain a boundary layer function having a rapidly oscillating character of change. In addition to such a boundary layer function, ordinary differential equations contain an exponential [1], and parabolic equations - parabolic [2, 3] and angular boundary layer [2, 5] functions. If the phase of the free term has stationary points, then boundary layers arise additionally, having a power character of change. In this case, the asymptotic solution consists of regular and boundary layer terms. The boundary layer members are parabolic, power, rapidly oscillating boundary layer functions, and their products, which are called angular boundary layer functions [4]. In this chapter we used the methods of [4, 5].

1.2 Statement of the problem

In this chapter we study the following problem:

$$L_\varepsilon u\ (x,t,\varepsilon) \equiv \partial_t u - \varepsilon^2 a(x)\partial_x^2 u - b(x,t)u = \sum_{k=1}^{N} f_\kappa(x,t)\exp\left(\frac{i\theta_\kappa(t)}{\varepsilon}\right), (x,t)\in\Omega, \tag{1}$$

$$u(x,t,\varepsilon)|_{t=0} = u(x,t,\varepsilon)|_{x=0} = u(x,t,\varepsilon)|_{x=1} = 0$$

where $\varepsilon > 0$ is a small parameter and $\Omega = \{(x, t): x\in(0,1), t\in(0,T]\}$.
The problem is solved under the following assumptions:

1. $a(x) > 0$, $a(x)\in C^\infty[0,1]$, $b(x,t)$, $f\ (x,t)\in C^\infty(\overline{\Omega})$.
2. $\forall x\in[0,1]$ function $a(x) > 0$.
3. $\theta_k'(t)\big|_{t=0} = 0$ is the phase function.

1.3 Regularization of the problem

For the regularization of problem (Eq. (1)), we introduce regularizing independent variables using methods [5, 6]:

$$\eta = \frac{t}{\varepsilon^2}, r_k = \frac{i\,[\theta_k(t)-\theta_k(0)]}{\varepsilon}, \xi_\nu = \frac{\varphi_\nu(x)}{\varepsilon}, i = \sqrt{-1},$$
$$\zeta_v = \frac{\varphi_\nu(x)}{\varepsilon^2}, \varphi_\nu(x) = (-1)^{\nu-1}\int_{\nu-1}^{x}\frac{ds}{\sqrt{a(s)}},\quad \nu = 1, 2, \tag{2}$$
$$\sigma_k = \int_0^t \exp\left(\frac{i\,[\theta_\kappa(s)-\theta_\kappa(0)]}{\varepsilon}\right)ds \equiv p_\kappa(t,\varepsilon), l = \overline{0,r}, j = \overline{0,k_l-1}$$

Instead of the desired function $u(x,t,\varepsilon)$, we will study the extended function

$\breve{u}(M,\varepsilon)$, $M = (x,t,r,\eta,\sigma,\xi,\zeta)$, $\sigma = (\sigma_1,\sigma_2 ... \sigma_N)$, $r = (r_1, r_2 ... r_N)$, $\xi = (\xi_1,\xi_2)$, $\zeta = (\zeta_1,\zeta_2)$

such that its restriction by regularizing variables coincides with the desired solution:

$$\breve{u}(M,\varepsilon)|_{\gamma=p(x,t,\varepsilon)} \equiv u(x,t,\varepsilon)$$
$$\gamma = (r,\sigma,\eta,\xi,\zeta) \tag{3}$$

Taking into account (Eqs. (21)) and ((3)), we find the derivatives
On the basis of (Eqs. (1)–(4)) for the extended function $\breve{u}(M,\varepsilon)$, we set the problem:

$$\partial_t u(x,t,\varepsilon) \equiv (\partial_t\breve{u}\ (M,\varepsilon) + \tfrac{1}{\varepsilon^2}\partial_\eta\breve{u}(M,\varepsilon) + \sum_{k=1}^{N}[\frac{i\theta_k{}'(t)}{\varepsilon}\partial_{r_k}\breve{u}(M,\varepsilon) + \exp(r_k)\partial_{\sigma_k}\breve{u}\ (M,\varepsilon))\Big|_{\gamma=p(x,t,\varepsilon)},$$
$$\partial_x u(x,t,\varepsilon) \equiv \left((\partial_x\breve{u}\ (M,\varepsilon) + \sum_{\nu=1}^{2}\left\{\frac{\varphi_\nu'(x)}{\varepsilon}\partial_{\xi_\nu}\breve{u}\ (M,\varepsilon) + \frac{\varphi_\nu'(x)}{\varepsilon^2}\partial_{\zeta_\nu}\breve{u}\ (M,\varepsilon)\right\}\right)\Big|_{\gamma=p(x,t,\varepsilon)}, \tag{4}$$

$$\partial_x^2 u(x,t,\varepsilon) \equiv \left(\left(\partial_x^2\, \check{u}\,(M,\varepsilon) + \sum_{\nu=1}^{2}\left\{\left(\frac{\varphi_\nu'(x)}{\varepsilon}\right)^2 \partial_{\xi_\nu}^2\, \check{u}\,(M,\varepsilon) + \left(\frac{\varphi_\nu'(x)}{\varepsilon^2}\right)^2 \partial_{\zeta_\nu}^2\, \check{u}\,(M,\varepsilon) + \frac{1}{\varepsilon}\, D_{\xi,v}\check{u}\,(M,\varepsilon) + \frac{1}{\varepsilon^2} D_{\zeta,v}\check{u}(M,\varepsilon)\right\}\right)\right|_{\gamma=p(x,t,\varepsilon)},$$

$$D_{\xi,v} \equiv 2\varphi_\nu'(x)\partial_{x,\xi_\nu}^2 + \varphi_\nu''(x)\partial_{\xi_\nu},$$
$$D_{\zeta,v} \equiv 2\varphi_\nu'(x)\partial_{x,\zeta_\nu}^2 + \varphi_\nu''(x)\partial_{\zeta_\nu}.$$

$$\begin{aligned}
\widetilde{L}_\varepsilon \check{u}(M,\varepsilon) &\equiv \frac{1}{\varepsilon^2}\, T_0\, \check{u}(M,\varepsilon) + \sum_{k=1}^{N} \frac{i\theta_k'(t)}{\varepsilon}\, \partial_{r_k}\check{u}(M,\varepsilon) + T_1\, \check{u}(M,\varepsilon) \\
&= \sum_{k=1}^{N} f_\kappa(x,t)\exp\left(r_k + \frac{i\theta_\kappa(0)}{\varepsilon}\right) + L_\zeta\, \check{u}(M,\varepsilon) + \varepsilon L_\xi \check{u}(M,\varepsilon) + \varepsilon^2 L_x \check{u}(M,\varepsilon) \\
&\quad \check{u}(M,\varepsilon)\big|_{t=r_k=\eta=0} = \check{u}(M,\varepsilon)\big|_{x=0,\,\xi_1=\zeta_1=0} = \check{u}(M,\varepsilon)\big|_{x=1,\,\xi_2=\zeta_2=0} = 0, \\
&T_1 \equiv \partial_\eta - \sum_{\nu=1}^{2}\partial_{\zeta_\nu}^2, \\
&T_2 \equiv \partial_t - \sum_{\nu=1}^{2}\partial_{\xi_\nu}^2 - b(x,t) + \sum_{k=1}^{N}\exp(r_k)\partial_{\sigma_k}, \\
&L_\xi \equiv a(x)\sum_{v=1}^{2} D_{\xi,v}, \\
&L_\zeta \equiv a(x)\sum_{v=1}^{2} D_{\zeta,v}, \\
&L_x \equiv a(x)\partial_x^2.
\end{aligned} \tag{5}$$

The problem (Eq. (5)) is regular in ε as $\varepsilon \to 0$:

$$\left(\widetilde{L}_\varepsilon \check{u}(M,\varepsilon)\right)\Big|_{q=q(x,t,\varepsilon)} \equiv L_\varepsilon \check{u}(x,t,\varepsilon). \tag{6}$$

1.4 Solution of iterative problems

The solution of problem (Eq. (5)) will be determined in the form of a series:

$$\check{u}(\,M,\varepsilon) = \sum_{v=0}^{\infty} \varepsilon^v u_v(M). \tag{7}$$

For the coefficients of this series, we obtain the following iterative problems:

$$\begin{aligned}
&T_1 u_0(M) = 0,\; T_1 u_1(M) = -i\sum_{k=1}^{N}\theta_k'(t)\partial_{r_k}u_0(M), \\
&T_1 u_2(M) == -i\sum_{k=1}^{N}\theta_k'(t)\partial_{r_k}u_1(M) - T_2 u_0(M) + \sum_{k=1}^{N} f_\kappa(x,t)\exp\left(r_k + \frac{i\theta_\kappa(0)}{\varepsilon}\right) + L_\zeta\, u_0(M), \\
&T_1 u_v(M) = -i\sum_{k=1}^{N}\theta_k'(t)\partial_{r_k}u_{v-1}(M) - T_2 u_{v-2}(M) + L_\zeta u_{v-2} + L_\xi u_{v-3}(M) + L_x u_{v-4}(M).
\end{aligned} \tag{8}$$

The solution of this problem contains parabolic boundary layer functions; internal power boundary layer functions which are connected with a rapidly oscillating free term in a phase which are vanished at $t = t_l$, $l = 0, 1, \ldots\ldots, n$ in addition; and the asymptotic also contain angular boundary layer functions. We introduce a class of functions in which the iterative problems will be solved:

$$G_0 \cong C^\infty\left(\overline{\Omega}\right),\ G_1 = \left\{u(M) : u(M) = \oplus_{l=1}^{2} G_0 \otimes \operatorname{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\right\},$$

$$G_2 = \left\{u(M) : u(M) = \oplus_{k=1}^{N} G_0 \otimes \exp\left(r_k\right)\right\},$$

$$G_3 = \left\{u(M) : u(M) = \oplus_{k=1}^{N} \oplus_{l=1}^{2} Y_k^l(N_l) \otimes \exp\left(r_k\right), \left\|Y_k^l(N_l)\right\| < c\ exp\left(-\frac{\varsigma_l^2}{8\eta}\right)\right\},$$

$$G_4 = \left\{u(M) : u(M) = \oplus_{k=1}^{N} G_0\left(\oplus_{l=1}^{2} G_0 \otimes \operatorname{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\right)\sigma_k\right\}, N_l = (x, t, \eta, \varsigma_1, \varsigma_2).$$

From these spaces we construct a new space:

$$G = \oplus_{l=0}^{4} G_l.$$

The element $u(M)\epsilon G$ has the form:

$$\begin{aligned} u(M) = {} & v(x,t) + \sum_{l=1}^{2} w^l(x,t) \operatorname{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right) \\ & + \sum_{k=1}^{N}\left[c_k(x,t) + \sum_{l=1}^{2} Y_k^l(N_l)\right] \exp\left(r_k\right) \\ & + \sum_{k=1}^{N}\left[z_k(x,t) + \sum_{l=1}^{2} q_k^l(x,t) \operatorname{erfc}\left(\frac{\xi_l^2}{2\sqrt{t}}\right)\right]\sigma_{k.} \end{aligned} \tag{9}$$

1.5 Solvability of intermediate tasks

The iterative problems (Eq. (9)) in general form will be written:

$$T_1 u(M) = H(M). \tag{10}$$

Theorem 1. Suppose that the conditions (1)–(3) and $H(M)\epsilon G_3$ are satisfied. Then, equation (Eq. (10)) is solvable in G.

Proof. Let the free term $H(M)\epsilon G_3$ be representable in the form:

$$H(M) = \sum_{k=1}^{N} \sum_{l=1}^{2} H_k^l(N_l), \left\|H_k^l(N_l)\right\| < c\ \exp\left(\frac{\varsigma_l^2}{8\eta}\right).$$

Then, by directly substituting function $u(M)\epsilon$ G from (Eq. (9)) in (Eq. (10)), we see that this function is a solution if and only if the function $Y_k^l(N_l)$ will be a solution of equation:

$$\partial_\eta Y_k^l(N_l) = \partial_{\varsigma_l}^2 Y_k^l(N_l) + H_k^l(N_l), l = 1, 2, k = 1, 2, \ldots, N. \tag{11}$$

With the corresponding boundary conditions, this equation has a solution which have the estimate:

$$\left\|Y_k^l(N_l)\right\| < c\ exp\left(\frac{\varsigma_l^2}{8\eta}\right).$$

The theorem is proven.

Theorem 2. Suppose that the conditions of *Theorem 1* are satisfied. Then, under additional conditions:

1.
$$u(M)\big|_{t=\eta=0} = 0, u(M)\big|_{x=l-1,\, \xi_l=0,\, \varsigma_l=0} = 0, l = 1, 2.$$

2.
$$L_\varsigma u(M) = 0, L_\xi u(M) = 0.$$

3.
$$i\sum_{k=1}^{N} \theta'_k(t)\partial_{r_k} u_v(M) + T_2 u_{v-1}(M) + h(M) \in G_3.$$

Eq. (10) is uniquely solvable.

Proof. By *Theorem 1* equation (Eq. (10)) has a solution that is representable in the form (Eq. (9)). With satisfying condition (1), we obtain

$$v(x,t)\Big|_{t=0} = -\sum_{k=1}^{N} c_k(x,0), w^l(x,t)\Big|_{t=0} = \overline{w}^l(x), \tag{12}$$

$$Y_k^l(N_l)\big|_{t=\eta=0} = 0, q_k^l(x,t)\big|_{t=0} = \overline{q}_k^l(x),\ \ d_k^l(x,t)\Big|_{t=0} = \overline{d}_k^l(x),$$

$$w^l(x,t)\big|_{x=l-1} = -c_k(l-1,t), q_k^l(x,t)\big|_{x=l-1} = -z_k(l-1,t), l = 1, 2.$$

Due to the fact that the function $\mathrm{erfc}\left(\frac{\theta}{2\sqrt{t}}\right)$ is zero at $\theta = 0$, the values for $w^l(x,t)\big|_{t=0}, q_k^l(x,t)\big|_{t=0}$ are chosen arbitrarily.

We calculate

$$\begin{aligned}
&i\sum_{k=1}^{N} \theta'_k(t)\partial_{r_k} u_v(M) + T_2 u_{v-1}(M) + h(M)\\
&= i\sum_{k=1}^{N} \theta'_k(t)\left[c_{k,v}(x,t) + \sum_{l=1}^{2} Y_{k,v}^l(N_l)\right]\exp(r_k) + [\partial_t v_{v-1}(x,t) - b(x,t)v_{v-1}(x,t)]\\
&+ \sum_{l=1}^{2}\left[\partial_t w_{v-1}^l(x,t) - b(x,t)w_{v-1}^l(x,t)\right]\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\\
&+ \sum_{k=1}^{N}\left[\partial_t c_{k,v-1}(x,t) - b(x,t)c_{k,v-1}(x,t) + \sum_{l=1}^{2}\left(\partial_t Y_{k,v-1}^l(N_l) - b(x,t)Y_{k,v-1}^l(N_l)\right)\right]\exp(\tau_k)\\
&+ \sum_{k=1}^{N}\left\{\partial_t z_{k,v-1}(x,t) - (x,t)z_{k,v-1}(x,t) + \sum_{l=1}^{2}\left[\partial_t q_{k,v-1}^l(x,t) - b(x,t)q_{k,v-1}^l(x,t)\right]\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\right\}\sigma_k\\
&+ \sum_{k=1}^{N}\left[z_{k,v-1}(x,t) + \sum_{l=1}^{2} q_{k,v-1}^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\right]\exp(\tau_k) + h_0(x,t) + \sum_{l=1}^{2} h_1^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\\
&+ \sum_{k=1}^{N}\left[h_2^k(x,t) + \sum_{l=1}^{2} h_2^{l,k}(x,t)\right]\exp(\tau_k) + \sum_{k=1}^{N}\left[h_3^k(x,t) + \sum_{l=1}^{2} h_3^{l,k}(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\right]\sigma_k.
\end{aligned} \tag{13}$$

Condition (3) of the theorem will be ensured, if we choose arbitrarily (Eq. (9)) as the solutions of the following equations:

$$\begin{gathered}\partial_t v_{v-1}(x,t) - b(x,t)v_{v-1}(x,t) = -h_0(x,t),\\ \partial_t w^l_{v-1}(x,t) - b(x,t)w^l_{v-1}(x,t) = -h^l_1(x,t),\\ \partial_t Y^l_{k,v-1}(N_l) - b(x,t)Y^l_{k,v-1}(N_l) = -\left(h^{l,k}_2(x,t) + q^l_{k,v-1}(x,t)\operatorname{erfc}\left(\frac{\varsigma_l}{2\sqrt{\eta}}\right)\right),\\ \partial_t z_{k,v-1}(x,t) - b(x,t)z_{k,v-1}(x,t) = -h^k_3(x,t),\\ \partial_t q^l_{k,v-1}(x,t) - b(x,t)q^l_{k,v-1}(x,t) = -h^{l,k}_3(x,t),\\ i\theta'_k(t)c_{k,v}(x,t) = -z_{k,v-1}(x,t) - [\partial_t c_{k,v-1}(x,t) - b(x,t)c_{k,v-1}(x,t)] - h^k_2(x,t).\end{gathered} \tag{14}$$

After this choice of arbitrariness, expression (Eq. (13)) is rewritten:

$$i\sum_{k=1}^{N}\theta'_k(t)\partial_{r_k}u_v(M) + T_2u_{v-1}(M) + h(M) = \sum_{k=1}^{N}\sum_{l=1}^{2}\left[i\theta'_k(t)Y^l_{k,v}(N_l)\right]\exp(\tau_k) \in G_3$$

In (Eq. (14)), transition was made from $\xi_l/2\sqrt{t}$ to variable $\varsigma_l/2\sqrt{\eta}$. The function $Y^l_k(N_l)$ is defined as the solution of equation (Eq. (30)) under the boundary conditions from (Eq. (12)) in the form:

$$Y^l_k(N_l) = d^l_k(x,t)\operatorname{erfc}\left(\frac{\varsigma_l}{2\sqrt{\eta}}\right) + \frac{1}{2\sqrt{\pi}}\int_0^\eta\int_0^\infty \frac{H^l_k(\cdot)}{\sqrt{\eta-\tau}}\left[\exp\left(-\frac{(\varsigma_l-y)^2}{4(\eta-\tau)}\right) - \exp\left(-\frac{(\varsigma_l+y)^2}{4(\eta-\tau)}\right)\right]dyd\tau. \tag{15}$$

We substitute this function in the corresponding equation from (Eq. (14)); then with respect to $d^l_k(x,t)$, we obtain a differential equation, which is solving under the initial condition $d^l_k(x,t)\Big|_{t=0} = \overline{d}^l_k(x)$, and we find

$$d^l_k(x,t) = \overline{d}^l_k(x,t)B(x,t) + P^l_k(x,t),\ B(x,t) = \exp\left(\int_0^t b(x,s)ds\right),$$

where $P^l_k(x,t)$ is known as the function.

By substituting the obtained function into condition for $d^l_k(x,t)\Big|_{x=l-1}$ from (Eq. (12)), we define the value of $\overline{d}^l_k(x)\Big|_{x=l-1}$. The obtained value is used as an initial condition for a differential equation with respect to $\overline{d}^l_k(x)$, which is obtained after substitution $d^l_k(x,t)$ into the first condition of (2). With that we ensure fulfillment of this condition and uniqueness of the function $Y^l_k(N_l)$. The last equation from (Eq. (14)) due to the fact that $\theta'_k(t_k) = 0$ is solvable if

$$z^l_{k,v-1}(x,0) = -h^k_2(x,0) - [\partial_t c_{k,v-1}(x,t) - b(x,t)c_{k,v-1}(x,t)]\Big|_{t=0}.$$

The obtained ratio is used as the initial condition for the differential equation with respect to $z^l_{k,v-1}(x,t)$ from (Eq. (14)).

The equation with respect to $v_{v-1}(x,t)$ under the initial condition from (12) determines this function uniquely. Equations with respect to $w^l_{k,v-1}(x,t)$, $q^l_{k,v-1}(x,t)$ under the corresponding condition from (Eq. (12)) have solutions representable in the form:

$$w^l_{k,\,v-1}(x,t) = \overline{w}^l_{k,\,v-1}(x)B(x,t) + H^l_{1,\,v-1}(x,t),$$
$$q^l_{k,\,v-1}(x,t) = \overline{q}^l_{k,\,v-1}(x)B(x,t) + H^l_{2,\,v-1}(x,t) \tag{16}$$

where $H^l_{1,\,v-1}(x,t)$, $H^l_{2,\,v-1}(x,t)$- are known functions.

With substituting (Eq. (16)) into the conditions under $x = l-1$ from (Eq. (12)), we define values of $\overline{w}^l_{k,\,v-1}(x)\big|_{x=l-1}, \overline{q}^l_{k,\,v-1}(x)\Big|_{x=l-1}$. These conditions are used in solving differential equations which are obtained from the second condition of (Eq. (21)):

$$L_\xi\left(w^l_{k,\,v-1}(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\right) = 0, L_\xi\left(q^l_{k,\,v-1}(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\right) = 0.$$

Thus, function $u(M)$ is determined uniquely. The theorem is proven.

1.6 Solution of iterative problems

Eq. (8) is homogeneous for k = 0; therefore, by *Theorem 1*, it has a solution in G, representable in the form:

$$u_0(M) = v_0(x,t) + \sum_{l=1}^{2} w^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)$$
$$+ \sum_{k=1}^{N}\left\{\left(c_{k,\,0}(x,t) + \sum_{l=1}^{2} Y^l_{k,\,0}(N_l)\right)e^{r_k} + \left[z_{k,\,0}(x,t) + \sum_{l=1}^{2} q^l_{k,\,0}(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)\right]\sigma_k\right\} \tag{17}$$

If the function $Y^l_{k,\,0}(N_l)$ is the solution of the equation $\partial_\eta Y^l_{k,\,0}(N_l) = \partial^2_{\varsigma_l} Y^l_{k,\,0}(N_l)$ which is satisfying that

$$Y^l_{k,\,0}(N_l)\big|_{t=\eta=0} = 0, Y^l_{k,\,0}(N_l)\big|_{x=l-1,\,\varsigma_l=0} = -c_{k,\,0}(l-1,t).$$

from the last problem, we define

$$Y^l_{k,\,0}(N_l) = d^l_{k,\,0}(x,t)\ \mathrm{erfc}\left(\frac{\varsigma_l}{2\sqrt{\eta}}\right), d^l_{k,\,0}(x,t)\big|_{x=l-1} = -c_{k,\,0}(l-1,t), \text{where } d^l_{k,\,0}(x,t)\Big|_{t=0} = \overline{d}^l_{k,\,0}(x).$$

$\overline{d}^l_{k,\,0}(x)$ is the arbitrary function. In the next step, equation (Eq. (8)) for k = 1 takes the form:

$$T_1u_1(M) = -i\sum_{k=1}^{N}\theta'_k(t)\left[c_{k,\,0}(x,t) + \sum_{l=1}^{2} Y^l_{k,\,0}(N_l)\right]e^{r_k}.$$

According to *Theorem 1*, this equation is solvable in U, if $c_{k,\,0}(x,t)$=0; the function $Y^l_{k,\,0}(N_l)$ is the solution of the differential equation $\partial_\eta Y^l_{k,\,0}(N_l) = \partial^2_{\varsigma_l} Y^l_{k,\,0}(N_l)+$ $H^l_{k,\,0}(N_l)$, and its solution is representable in the form (Eq. (14)), where $H^l_k(0) = i\theta'_k(t)Y^l_{k,\,0}(N_l)$. Satisfying condition (1)–(3) of *Theorem 1*, we obtain (see (Eq. (14)))

$$
\begin{gathered}
\partial_t v_0 - b(x,t)v_0(x,t) = 0,\ \partial_t w_0^l(x,t) - b(x,t)w_0^l(x,t) = 0, \\
\partial_t d_{k,0}^l(x,t) - b(x,t)d_{k,0}^l(x,t) = -q_{k,0}^l(x,t), \\
\partial_t z_{k,0}(x,t) - b(x,t)z_{k,0}(x,t) = 0, \\
\partial_t q_{k,0}^l(x,t) - b(x,t)q_{k,0}^l(x,t) = 0, \\
i\theta_k'(t)c_{k,1}(x,t) = -z_{k,0}(x,t) + f_k(x,t)\exp\left(\frac{i\theta_\kappa(0)}{\varepsilon}\right), \\
L_\varsigma\left(d_{k,0}^l(x,t)\operatorname{erfc}\left(\frac{\varsigma_l}{2\sqrt{\eta}}\right)\right) = 0.
\end{gathered}
\tag{18}
$$

When the equation is obtained with respect to $d_{k,0}^l(x,t)$ in the $q_{k,0}^l(x,t)\operatorname{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right)$, a transition $\frac{\xi_l}{2\sqrt{t}} = \frac{\varsigma_l}{2\sqrt{\eta}}$ occurs.

The initial conditions for equation (Eq. (18)) are determined from (Eq. (12)). Functions $w_0^l(x,t)$, $d_{k,0}^l(x,t)$, $q_{k,0}^l(x,t)$ are expressed through arbitrary functions $\overline{w}_0^l(x)$, $\overline{d}_{k,0}^l(x)$, $\overline{q}_{k,0}^l(x)$. These arbitrary functions provide the condition:

$$
L_\xi u_k(m) = 0,\ L_\varsigma u_k(m) = 0,
$$

ensuring the solvability of the equation with respect to $c_{k,1}^l(x,t)$. Suppose that

$$
Z_{k,0}(x,t)\big|_{t=0} = f_k(x,t)\exp\left(\frac{i\theta_k(0)}{\varepsilon}\right).
$$

This relation is used by the initial condition for determining $Z_{k,0}(x,t)$ from the equation entering into (Eq. (18)).

Further repeating this process, we can determine all the coefficients of $u_k(m)$ of the partial sum:

$$
u_{\varepsilon n}(m) = \sum_{i=0}^{n} \varepsilon^i u_i(m).
$$

In each iteration with respect to $v_i(x,t)$, $w_i^l(x,t)$, $d_{k,i}^l(x,t)$, $z_{k,i}(x,t)$, $q_{k,i}^l(x,t)$, we obtain inhomogeneous equations.

1.7 Assessment of the remainder term

For the remainder term

$$
R_{\varepsilon n}(x,t,\varepsilon) \equiv R_{\varepsilon n}(m,\varepsilon)\Big|_{\gamma=\rho(x,t,\varepsilon)} = u(x,t,\varepsilon) - \sum_{i=0}^{n}\varepsilon^i u_i(m)\Big|_{\gamma=\rho(x,t,\varepsilon)},
$$

taking into account (Eqs. (3) and (6)), we obtain the equation

$$
L_\varepsilon R_{\varepsilon n}(x,t,\varepsilon) = \varepsilon^{n+1} g_n(x,t,\varepsilon)
$$

with homogeneous boundary conditions. Using the maximum principle, like work of [7], we get the estimate:

$$
|R_{\varepsilon n}(x,t,\varepsilon)| < c\varepsilon^{n+1}. \tag{19}
$$

Theorem 3. Suppose that conditions (1)–(3) are satisfied. Then, the constructed solution is an asymptotic solution of problem (Eq. (1)), i.e., $\forall n = 0, 1, 2, \ldots$; the estimate is fair (Eq. (18)).

2. Two-dimensional parabolic problem with a rapidly oscillating free term

2.1 Introduction

In the case when a small parameter is also included as a multiplier with a temporal derivative, the asymptotic of the solution acquires a complex structure.

Different classes of singularly perturbed parabolic equations are studied in [2]. There, regularized asymptotics of the solution of these equations are constructed, when a small parameter is in front of the time derivative and with one spatial derivative. It is shown that the constructed asymptotic contains exponential, parabolic, and angular products of exponential and parabolic boundary layer functions. The equations are studied when the limiting equation has a regular singularity. Such equations have a power boundary layer. If a small parameter is entering as the multiplier for all spatial derivatives, then the asymptotic solution contains a multidimensional parabolic boundary layer function. When entering into the equation, as free terms of rapidly oscillating functions, then the asymptotic of the solution additionally contains fast-oscillating boundary layer functions. If it is additionally assumed that the phase of this free term has a stationary point, in addition to the rapidly oscillating boundary layer function that arises as a power boundary layer.

This section is devoted to a two-dimensional equation of parabolic type.

2.2 Statement of the problem

Consider the problem:

$$L_\varepsilon u(x,t,\varepsilon) \equiv \partial_t u - \varepsilon^2 \Delta_a u - b(x,t)u = f(x,t)\exp\left(\frac{i\theta(t)}{\varepsilon}\right), (x,t)\epsilon E,$$
$$u|_{t=0} = 0, u|_{\partial\Omega=0} = 0, \tag{20}$$

where $\varepsilon > 0$ is the small parameter, $x = (x_1, x_2)$, $\Omega = (0 < x_1 < 1)x$ $(0 < x_2 < 1)$, $E = (0 < t \le T)x\Omega$, $\Delta_a \equiv \sum_{l=1}^{2} a_l(x_l)\partial_{x_l}^2$.

The problem is solved under the following assumptions:

1. $\forall x_l \in [0,1]$ the function $a_l(x_l) \in C^\infty[0,1]$, $l = 1, 2$.

2. $b(x,t), f(x,t) \in C^\infty[E]$.

3. $\theta'(0) = 0$.

2.3 Regularization of the problem

Following the method of regularization of singularly perturbed problems [1, 2], along with the independent variables (x, t), we introduce regularizing variables:

$$\mu = \frac{t}{\varepsilon}, \xi_l = \frac{(-1)^{l-1}}{\sqrt{\varepsilon^3}} \int_{l-1}^{x_1} \frac{ds}{\sqrt{a_1(s)}}, \eta_l = \frac{\varphi_l(x_1)}{\varepsilon^2}$$
$$\xi_{l+2} = \frac{(-1)^{l-1}}{\sqrt{\varepsilon^3}} \int_{l-1}^{x_2} \frac{ds}{\sqrt{a_2(s)}}, \eta_{l+2} = \frac{\varphi_{l+2}(x_2)}{\varepsilon^3} \tag{21}$$
$$\sigma = \int_0^t e^{\frac{i[\theta(s)-\theta(0)]}{\varepsilon}} ds, \tau_2 = \frac{i[\theta(t) - \theta(0)]}{\varepsilon}, \tau_1 = \frac{t}{\varepsilon^2},$$
$$\varphi_l(x_r) = (-1)^{l-1} \int_{l-1}^{x_r} \frac{ds}{\sqrt{a_r(s)}},$$

For extended function $\widetilde{u}(M, \varepsilon), M = (x, t, \tau, \xi, \eta)$ such that

$$\widetilde{u}(M, \varepsilon)|_{\mu=\psi(x,t,\varepsilon)} \equiv u(x, t, \varepsilon),$$
$$\chi = (\tau, \xi, \eta), \tau = (\tau_1, \tau_2), \xi = (\xi_1, \xi_2, \xi_3, \xi_4),$$
$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4),$$
$$\psi(x, t, \varepsilon) = \left(\frac{t}{\varepsilon^2}, \frac{t}{\varepsilon}, \frac{i[\theta(t) - \theta(0)]}{\varepsilon}, \frac{\varphi(x)}{\varepsilon}, \frac{\varphi(x)}{\varepsilon^2} \right),$$
$$\varphi(x) = (\varphi_1(x_1), \varphi_2(x_1), \varphi_3(x_2), \varphi_4(x_2))$$
$$\widetilde{u}(M, \varepsilon)|_{\mu=\psi(x,t,\varepsilon)} \equiv u(x, t, \varepsilon), \chi = (\tau, \xi, \eta), \tag{22}$$
$$\tau = (\tau_1, \tau_2), \xi = (\xi_1, \xi_2, \xi_3, \xi_4),$$
$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4),$$
$$\psi(x, t, \varepsilon) = \left(\frac{t}{\varepsilon^2}, \frac{t}{\varepsilon}, \frac{i[\theta(t) - \theta(0)]}{\varepsilon}, \frac{\varphi(x)}{\varepsilon}, \frac{\varphi(x)}{\varepsilon^2} \right),$$
$$\varphi(x) = (\varphi_1(x_1), \varphi_2(x_1), \varphi_3(x_2), \varphi_4(x_2)).$$

Find from (Eq. (22)) the derivatives based on

$$\partial_t u \equiv \left(\partial_t \widetilde{u} + \tfrac{1}{\varepsilon} \partial_\mu \widetilde{u} + \tfrac{1}{\varepsilon^2} \partial_{\tau_1} \widetilde{u} + \tfrac{i\theta'(t)}{\varepsilon} \partial_{\tau_2} \widetilde{u} + \exp(\tau_2) \partial_\sigma \widetilde{u} \right) \Big|_{\chi=\psi(x,t,\varepsilon)},$$
$$\partial_{x_r} u \equiv \left(\partial_{x_r} \widetilde{u} + \sum_{l=2r-1}^{2r} \left[\frac{\varphi_l'(x_r)}{\sqrt{\varepsilon^3}} \partial_{\xi_l} \widetilde{u} + \frac{\varphi_l'(x_r)}{\varepsilon^2} \partial_{\zeta_l} \widetilde{u} \right] \right) \Big|_{\chi=\psi(x,t,\varepsilon)},$$
$$\partial_{x_r}^2 u \equiv \left(\partial_{x_r}^2 \widetilde{u} + \sum_{l=2r-1}^{2r} \left[\frac{\varphi_l'2(x_r)}{\varepsilon^3} \partial_{\xi_l}^2 \widetilde{u} + \frac{\varphi_l'2(x_r)}{\varepsilon^4} \partial_{\zeta_l}^2 \widetilde{u} \right] \right.$$
$$\left. + \sum_{l=2r-1}^{2r} \left[\frac{2\varphi_l'(x_r)}{\sqrt{\varepsilon^3}} \partial_{x_r \xi_l}^2 \widetilde{u} + \frac{\varphi_l''(x_r)}{\sqrt{\varepsilon^3}} \partial_{\xi_l} \widetilde{u} + \frac{1}{\varepsilon^2} \left(\varphi_l'(x_r) \partial_{x_r \eta_l}^2 \widetilde{u} + \varphi_l''(x_r) \partial_{\eta_l} \widetilde{u} \right) \right] \right) \Big|_{\chi=\psi(x,t,\varepsilon)} \tag{23}$$

Below, it is shown that the solution of the iterative problems does not contain terms depending on (ξ_1, ξ_2), (ξ_3, ξ_4), (ζ_1, ζ_2), (ζ_3, ζ_4), (ξ_l, ζ_k), $l, k = 1, 2$. Therefore, to simplify recording, the mixed derivatives of these variables are omitted. Based on (Eq. (20)), (Eq. (22)), and (Eq. (23)) for extended function $\widetilde{u}(M, \varepsilon)$, set the problem:

$$\widetilde{L}_\varepsilon \widetilde{u} \equiv \frac{1}{\varepsilon^2} T_0 \widetilde{u} + \frac{1}{\varepsilon} i\theta'(t)\partial_{\tau_2}\widetilde{u} + \frac{1}{\varepsilon} T_1 \widetilde{u} + D_\sigma \widetilde{u} - L_\eta \widetilde{u} - \sqrt{\varepsilon} L_\xi \widetilde{u} - \varepsilon^2 \Delta_a \widetilde{u} = f(x,t)\exp\left(\tau_2 + \frac{i\theta(0)}{\varepsilon}\right),$$
$$\widetilde{u}|_{t=\tau_1=\tau_2=0} = 0, \widetilde{u}|_{x_l=r-1,\,\xi_k=\eta_k=0} = 0, r = 1, 2, \quad l = 1, 2, k = \overline{1, 4}$$
$$T_0 \equiv \partial_{\tau_1} - \Delta_\eta, T_1 \equiv \partial_\mu + \Delta_\xi, D_\sigma \equiv D_t + \exp(\tau_2)\partial_\sigma, D_t \equiv \partial_t + b(x,t),$$
$$L_\eta \equiv \sum_{r=1}^{2} \sum_{l=2r-1}^{2r} a_r(x_r) D_{x,\eta}^{r,l},$$
$$D_{x,\xi}^{r,l} \equiv \left[2\varphi_l'(x_r)\partial^2_{x_r\xi_l} + \varphi_l''(x_r)\partial_{\eta_l}\right],$$
$$\Delta_\eta \equiv \sum_{k=1}^{4} \partial^2_{\eta_k}, \quad E_1 = E\mathrm{x}(0,\infty)^{10} \tag{24}$$

In this case, the identity is satisfied:

$$\left(\widetilde{L}_\varepsilon \widetilde{u}\right)\Big|_{\chi=\psi(x,t,\varepsilon)} \equiv L_\varepsilon u(x,t,\varepsilon). \tag{25}$$

2.4 Solution of iterative problems

For the solution of the extended function (Eq. (24)), we search in the form of series

$$\widetilde{u}(M,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^{\frac{i}{2}} u_i(M). \tag{26}$$

Then, for the coefficients of this series, we get the following problems:

$$\begin{aligned}
&T_0 u_v(M) = 0, v = 0, 1,\\
&T_0 u_q = -i\theta'(t)\partial_{\tau_2} u_{q-2} - T_1 u_{q-2}, q = 2, 3.\\
&T_0 u_4 = f(x,t)\exp\left(\tau_2 + \frac{i\theta(0)}{\varepsilon}\right) - T_1 u_2 - D_\sigma u_0 + L_\eta u_0,\\
&T_0 u_i = -i\theta'(t)\partial_{\tau_2} u_{i-2} - T_1 u_{i-2} - D_\sigma u_{i-4} + L_\eta u_{i-4} + L_\xi u_{i-5} + \Delta_a u_{i-8},\\
&u_i|_{t=\tau=0} = 0, u_i|_{x_l=r-1,\,\xi_k=\eta_k=0} = 0, l, r = 1, 2.k = \overline{1, 4}
\end{aligned} \tag{27}$$

We introduce a class of functions:

$$U_0 = \left\{V_0(N) = [c(x,t) + F_1(N) + F_2(N)]\exp(\tau_2), F_1(N) \in U_4, F_2(N) \in U_5, c(x,t) \in C^\infty\left(\overline{E}\right)\right\},$$
$$U_1 = \left\{V_1(M) : V_1(M) = v(x,t) + F_1(M) + F_2(M), F_1(M) \in U_4, F_2(M) \in U_5, v(x,t) \in C^\infty\left(\overline{E}\right)\right\},$$
$$U_2 = \left\{V_2(M) : V_2(M) = [z(x,t) + F_1(M) + F_2(M)]\sigma, F_1(M) \in U_4, F_2(M) \in U_5, z(x,t) \in C^\infty\left(\overline{E}\right)\right\},$$
$$U_4 = \left\{V_4(M) : V_1(M) = \sum_{l=1}^{4} Y^l(N_l), \left|Y^l(N_l)\right| < c\exp\left(-\frac{\eta_l^2}{8\tau_1}\right)\right\},$$
$$U_5 = \left\{V_5(M) : V_2(M) = \sum_{r,\,l=1}^{4} Y^{r+2,l}(N_{r+2,l}), \left|Y^{r+2,l}(N_{r+2,l})\right| < c\exp\left(-\frac{\left|\eta^{r,l}\right|^2}{8\tau_1}\right), \left|\eta^{r,l}\right| = \sqrt{\eta_r^2 + \eta_l^2}\right\}.$$

From these classes we will construct a new one, as a direct sum:

$$U = U_0 \bigoplus U_1 \bigoplus U_2.$$

Any item $u(M) \in U$ is representable in the form:

$$u(M) = v(x,t) + c(x,t)\exp(\tau_2) + z(x,t)\sigma + \left[\sum_{l=1}^{4} Y^l(N_l) + \sum_{r,\,l=1}^{2} Y^{r+2,\,l}(N_{r+2,\,l})\right]\exp(\tau_2)$$

$$+\sum_{l=1}^{4} w^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{l,\,r=1}^{2} w^{r+2,\,l}(M_{r+2,\,l}) + \left[\sum_{l=1}^{4} q^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{l,\,r=1}^{2} z^{r+2,\,l}(M_{r+2,\,l})\right]\sigma,$$

$$N_l = (x,t,\tau_1,\eta_l), N_{r+2,\,l} = (x,t,\tau_1,\eta_l,\eta_{r+2}),$$

$$M_l = (x,t,\mu,\xi_l), M_{r+2,\,l} = (x,t,\mu,\xi_l,\xi_{r+2}). \tag{28}$$

Let's satisfy this function to the boundary conditions:

$$v(x,0) = -c(x,0), Y^l(N_l)\big|_{t=\tau_1=0} = 0 \tag{29}$$

$$Y^{r+2,\,l}(N_{r+2,\,l})\big|_{t=\tau_1=0} = 0, w^l\big|_{t=0} = \overline{w}^l(x),$$

$$q^l\big|_{t=0} = \overline{q}^l(x),$$

$$w^{r+2,\,l}(M_{r+2,\,l})\big|_{t=\mu=0} = 0,$$

$$z^{r+2,\,l}(M_{r+2,\,l})\big|_{t=\mu=0} = 0,$$

$$w^l(x,t)\big|_{x_1=l-1} = -v(l-1,x_2,t),$$

$$q^l(x,t)\big|_{x_1=l-1} = -z(l-1,x_2,t),$$

$$Y^l\big|_{x_1=l-1,\,\eta_l=0} = -c(l-1,x_2,t),\ \ Y^{r+2,\,l}\big|_{x_1=l-1,\,\eta_l=0} = -Y^{r+2,\,l}(N_{r+2,\,l})\big|_{x_1=l-1},$$

$$w^{r+2,\,l}\big|_{x_1=l-1,\,\xi_l=0} = -w^{r+2}(l-1,x_2,t)\mathrm{erfc}\left(\frac{\xi_{r+2}}{2\sqrt{t}}\right),$$

$$z^{r+2,\,l}\big|_{x_1=l-1,\,\xi_l=0} = -q^{r+2}(l-1,x_2,t)\mathrm{erfc}\left(\frac{\xi_{r+2}}{2\sqrt{t}}\right),$$

$$w^l(x,t)|_{x_r=l-1} = -v(x,t)|_{x_r=l-1},$$

$$q^l(x,t)|_{x_r=l-1} = -z(x,t)|_{x_r=l-1},$$

$$Y^{r+2}\big|_{x_2=l-1,\,\eta_{r+2}=0} = -c(x_1,l-1,t),$$

$$Y^{r+2,\,l}\big|_{x_2=l-1,\,\eta_{r+2}=0} = -Y^l\big|_{x_2=l-1},$$

$$w^{r+2,\,l}\big|_{x_2=l-1,\,\xi_{r+2}=0} = -w^l\big|_{x_2=l-1}\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right),$$

$$z^{r+2,\,l}\big|_{x_2=l-1,\,\xi_{r+2}=0} = -q^l\big|_{x_2=l-1}\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{t}}\right), l, r = 1, 2.$$

We compute the action of the operators T_0, T_1, L_η, L_ξ on function $u(M) \in U$, and we have

$$T_1 u(M) = \sum_{r,\,l=1}^{2}\left\{\partial_\mu w^{r+2,\,l} - \Delta_\xi w^{r+2,\,l} + \sigma\left[\partial_\mu z^{r+2,\,l} - \Delta_\xi z^{r+2,\,l}\right]\right\},$$

$$L_\eta u = \sum_{r=1}^{2}\sum_{l=2r-1}^{2r} D_{x,\eta}^{r,l} Y^l(N_l) + \sum_{v=1r,}^{2}\sum_{l=1}^{2} D_{x,\eta}^{v,l} Y^{r+2,l}(N_{r+2,l}),$$

$$L_\xi u = \sum_{r=1}^{2}\sum_{l=2r-1}^{2r} D_{x,\xi}^{r,l} w^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{v=1r,}^{2}\sum_{l=1}^{2} D_{x,\xi}^{v,l} w^{r+2,l}(M_{r+2,l})$$

$$+\sigma\left[\sum_{r=1}^{2}\sum_{l=2r-1}^{2r} D_{x,\xi}^{r,l} q^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{v=1r,}^{2}\sum_{l=1}^{2} D_{x,\xi}^{v,l} z^{r+2,l}(M_{r+2,l})\right],$$

$$D_\sigma u(M) = D_t v(x,t) + \sum_{l=1}^{4} D_t w^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{r,\,l=1}^{2} D_t w^{r+2,l}(M_{r+2,l})$$

$$+\left[D_t c(x,t) + \sum_{l=1}^{4} D_t Y^l(N_l) + \sum_{r,\,l=1}^{2} D_t Y^{r+2,l}(N_{r+2,l})\right]\exp(\tau_2)$$

$$+\sigma\left[D_t z(x,t) + \sum_{l=1}^{4} D_t q^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{r,\,l=1}^{2} D_t z^{r+2,l}(M_{r+2,l})\right]$$

$$+\left[z(x,t) + \sum_{l=1}^{4} q^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{r,\,l=1}^{2} z^{r+2,l}(M_{r+2,l})\right]\exp(\tau_2)$$

We write iterative equation (8) in the form:

$$T_0 u(M) = H(M). \tag{31}$$

Theorem 1. Let be $H(M) \in U_4 \bigoplus U_5$ and condition (1) is satisfied. Then, Eq. (31) is solvable in U, if the equations are solvable:

$$T_0 Y^l(N_l) = H_1(N_l),\, l = \overline{1,4},\, T_0 Y^{r+2,l}(N_{r+2,l}) = H_2(N_{r+2,l}),\, r, l = 1, 2.$$

Theorem 2. Let be $H_1(N_l) \in U_4$. Then, the problem $\partial_{\tau_1} Y^l(N_l) = \Delta_\eta Y^l(N_l) + H_1(N_l)$, $Y^l(N_l)\big|_{\tau_1=0} = 0$, $Y^l(N_l)\big|_{\eta_l=0} = d^l(x,t)$, $l = \overline{1,4}$ (Eq. (32)) has a solution $Y^l(N_l) \in U_4$.

Theorem 3. Let be $H_2(N_{r+2,l}) \in U_5$, $Y^l(N_l) \in U_4$, and then the problem $\partial_{\tau_1} Y^{r+2,l}(N_{r+2,l}) = \Delta_\eta Y^{r+2,l}(N_{r+2,l}) + H_2(N_{r+2,l})$, $Y^{r+2,l}(N_{r+2,l})\big|_{\eta_l=0} = -Y^{r+2}(N_{r+2})$, $Y^{r+2,l}(N_{r+2,l})\big|_{\eta_{r+2}=0} = -Y^l(N_l)$, $r, l = 1, 2$ has a solution $Y^{r+2,l}(N_{r+2,l}) \in U_5$.

The proof of these theorems is given in [2].

2.5 The decision of the iterative problems

Eq. (27) under $v = 0, 1$ is homogeneous. By *Theorem 1*, it has a solution representable in the form $u_0(M) \in U$ if functions $Y^l(N_l)$ and $Y^{r+2,l}(N_{r+2,l})$ – are solutions of the following equations:

$$T_0 Y_v^l(N_l) = 0,\; T_0 Y_v^{r+2,l}(N_{r+2,l}) = 0.$$

Based on the boundary conditions from (Eq. (29)), the solution is written:

$$Y_v^l(N_l) = d_v^l(x,t)\mathrm{erfc}\left(\frac{\eta_l}{2\sqrt{\tau_1}}\right),\, l = 1, 2, 3, 4. \tag{32}$$

$$Y_v^{r+2,l}(N_{r+2,l}) = -\int_0^{\tau_1}\int_0^\infty Y_v^l(*)\left[\frac{\partial}{\partial_\xi}\;G(N_l,\xi,\eta,\tau_1-\tau\,)\right]|_{\xi=0}d\eta d\tau$$
$$-\int_0^t\int_0^\infty Y_v^{r+2}(*)\left[\frac{\partial}{\partial_\eta}\;G(N_{r+2,l},\xi,\eta,\tau_1-\tau\,)\right]|_{\eta=0}d\xi d\tau,$$

where $d^l(x,t)$– is arbitrary function such as

$$d_v^p(x,t)\Big|_{t=0} = -\overline{d}_v^p(x), d_v^l(x,t)\Big|_{x_1=l-1} = -c_v(l-1,x_2,t),$$
$$G(\eta_l,\eta_{r+2,l},\xi,\eta,\tau_1) = \frac{1}{4\pi\tau_1}\left\{\exp\left(-\frac{(\eta_l-\xi)^2}{4\tau_1}\right) - \exp\left(-\frac{(\eta_l+\xi)^2}{4\tau_1}\right)\right\} \quad (33)$$
$$\left\{\exp\left(-\frac{(\eta_{r+2}-\eta)^2}{4\tau_1}\right) - \exp\left(-\frac{(\eta_{r+2}+\eta)^2}{4\tau_1}\right)\right\}.$$

Due to the fact that the function $d_v^l(x,t)$ при $t=\tau_1=0$ multiplied by the function becomes as $d_0^l(x,t)\Big|_{t=0} = -\overline{d}_0^l(x)$, an arbitrary function is accepted, and its values under $x_1 = l-1$ are determined from the second relation. According to *Theorems 2* and *3*, the functions found by the formula (Eq. (33)) satisfy the estimates:

$$\left|Y_v^l(N_l)\right| < cexp\left(-\frac{\eta_l^2}{8\tau_1}\right), \left|Y_v^{r+2,l}(N_{r+2,l})\right| < cexp\left(-\frac{\eta_{r+2}^2+\eta_l^2}{8\tau_1}\right), r,l=1,2. \quad (34)$$

Free member of equation (Eq. (27)) under $v = 2, 3$ has a form

$$F_{v-2}(M) \equiv T_1u_{v-2}(M) + i\theta'(t)\partial_\sigma u_{v-2}(M) = i\theta'(t)\left[c_{v-2}(x,t) + \sum_{l=1}^{4}Y_{v-2}^l(N_l) + \sum_{r,\,l=1}^{2}Y_{v-2}^{r+2,l}(N_{r+2,l})\right]$$
$$\exp(\tau_2) + \sum_{l,\,r=1}^{2}\left\{\partial_\mu w_{v-2}^{r+2,l} - \Delta_\xi w_{v-2}^{r+2,l} + \sigma\left[\partial_\mu z_{v-2}^{r+2,l} - \Delta_\xi z_{v-2}^{r+2,l}\right]\right\},$$

so that equation (Eq. (27)), under $v = 2, 3$, has a solution in U; we set

$$c_{v-2}(x,t) = 0, T_1w_{v-2}^{r+2,l} = 0, T_1z_{v-2}^{r+2,l} = 0.$$

Solutions of the last equations under the boundary conditions from (Eq. (29)) have a form (Eq. (33)) for which estimates of the form (Eq. (35) are fair. Eq. (27), i=4, has a free term:

$$F_4(M) = -i\theta'(t)\partial_{\tau_2} - T_1u_2 + f(x,t)\exp\left(\frac{i\theta(0)}{\varepsilon}\right) - D_\sigma u_0 + L_\eta u_0$$
$$= -i\theta'(t)\left[c_2(x,t) + \sum_{l=1}^{4}Y_2^l(N_l) + \sum_{r,\,l=1}^{2}Y_2^{r+2,l}(N_{r+2,l})\right]\exp(\tau_2)$$
$$-\sum_{l,\,r=1}^{2}\left[T_0w_2^{r+2,l}(M_{r+2,l}) + \sigma T_0z_2^{r+2,l}\right] - D_tv_0(x,t) - \sum_{l=1}^{4}D_tw_0^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right)$$
$$-\sum_{l,\,r=1}^{2}D_tw_0^{r+2,l}(x,t) - \exp(\tau_2)\left[\partial_tc_0(x,t) + \sum_{l=1}^{4}\partial_tY_0^l + \sum_{l,\,r=1}^{2}D_tY_0^{r+2,l}\right]$$

$$-\sigma\left[D_t z_0(x,t) + \sum_{l=1}^{4} D_t q_0^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{r,\,l=1}^{2} D_t z_0^{r+2,\,l}(M_{r+2,\,l})\right]$$
$$-\left[z_0(x,t) + \sum_{l=1}^{4} q_0^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{r,\,l=1}^{2} z_0^{r+2,\,l}(M_{r+2,\,l})\right]\exp(\tau_2)$$
$$+\sum_{r=1}^{2}\sum_{l=2r-1}^{2r} D_{x,\,\xi}^{r,\,l} w_0^p(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right) + \sum_{v=1}^{2}\sum_{r,\,l=1}^{2} D_{x,\,\eta}^{v,\,l} Y_0^{r+2,\,l}(N_{r+2,\,l}).$$

By providing $F_4(M) \in U_4 \bigoplus U_5$ with regard to $c_v(x,t) = 0$, $v = 0, 1$, we set

$$\begin{gathered} -i\theta'(t)c_2(x,t) + f(x,t)\exp\left(\frac{i\theta(0)}{\varepsilon}\right) - z_0(x,t) = 0, \\ D_t v_0(x,t) = 0,\ D_t z_0(x,t) = 0, \\ D_t Y_0^l(N_l),\ T_0 w_2^{r+2,\,l} = 0,\ T_0 z_2^{r+2,\,l} = 0, \\ D_t w_0^l = 0,\ D_t w_0^{r+2,\,l} = 0,\ D_t Y_0^{r+2,\,l} = 0, \\ D_t q_0^l\ (x,t) = 0,\ D_t z_0^{r+2,\,l}\ (x,t) = 0, \\ D_{x,\,\xi}^{r,\,l} w_0^l(x,t) = 0,\ D_{x,\,\eta}^{v,\,l} Y_0^{r+2,\,l} = 0,\ D_{x,\,\eta}^{r,\,l} Y_0^l = 0, \end{gathered} \tag{35}$$

then

$$F_4(M) = -i\theta'(t)\left[\sum_{l=1}^{4} Y_2^l(N_l) + \sum_{r,\,l=1}^{2} Y_2^{r+2,\,l}(N_{r+2,\,l})\right]\exp(\tau_2)$$
$$-\left[\sum_{l=1}^{4} q_0^l(x,t)\mathrm{erfc}\left(\frac{\eta_l}{2\sqrt{\tau_2}}\right) + \sum_{r,\,l=1}^{2} z_0^{r+2,\,l}(N_{r+2,\,l})\right]\exp(\tau_2).$$

In the last bracket, the transition is from the variables $\frac{\xi_l}{2\sqrt{\mu}}$ to the variables $\frac{\eta_l}{2\sqrt{\tau_2}}$.

Substituting the value $Y_0^l(N_l) = d_0^l(x,t)\mathrm{erfc}\left(\frac{\eta_l}{2\sqrt{\tau_1}}\right)$ into equation $D_t Y_0^l(N_l) = 0$, with respect to $d_0^l(x,t)$, we get the equation $D_t d_0^l(x,t) = 0$, which is solved under an arbitrary initial condition $d_0^l(x,t)\Big|_{t=0} = \overline{d}_0^l(x)$. This arbitrary function provides the condition $L_\eta\ Y_0^l = 0$; therefore, $D_{x,\,\eta}\ Y_0^l = 0$.The initial condition for this equation is determined from the relation:

$$d_0^l(x,t)\Big|_{x_1=l-1} = -c_0(l-1, x_2, t),\ d_0^{l+2}(x,t)\Big|_{x_2=l-1} = -c_0(x_1, l-1, t),$$

which comes out from (Eq. (29)) and (Eq. (33)). The function $Y_0^{r+2,\,l}(N_{r+2,\,l})$ expresses through $Y_0^l(N_l)$ therefore provided that

$$D_t Y_0^{r+2,\,l} = 0,\ D_{x,\,\eta}^{v,\,l} Y_0^{r+2,\,l} = 0.$$

The same is true for functions $w_0^{r+2,\,l}(M_{r+2,\,l}), z_0^{r+2,\,l}(M_{r+2,\,l})$; in other words, the following relations hold: $D_t w_0^{r+2,\,l} = 0$, $D_t z_0^{r+2,\,l} = 0$, $D_{x,\,\xi}^{v,\,l} w_0^{r+2,\,l} = 0$, $D_{x,\,\xi}^{v,\,l} z_0^{r+2,\,l} = 0$.

Solutions of equations with respect $w_0^{r+2,\,l}, z_0^{r+2,\,l}$ under appropriate boundary conditions from (Eq. (29) are representable as (Eq. (33)), and they are expressed through $w_2^l(x,t), q_2^l(x,t)$. The first equation (Eq. (36)) is solvable, if $z_0(x,t)|_{t=0} = f(x,0)\exp\left(\frac{i\theta(0)}{\varepsilon}\right)$. This ratio is used by the initial condition for the equation

$D_t z_0(x,t) = 0$. The remaining equations from (Eq. (36)) are solvable under the initial conditions from (Eq. (29)).

Thus, the main term of the asymptotics is uniquely determined. As can be seen from the representation (Eq. (28)) and the estimates (Eq. (35)), we note that the asymptotics of the solution have a complex structure. In addition to regular members, it contains various boundary layer functions. Parabolic boundary layer functions have an estimate:

$$\left|Y^l(N_l)\right| < cexp\left(-\frac{\eta_l^2}{8\tau_1}\right),\ \left|w^l(x,t)\mathrm{erfc}\left(\frac{\xi_l}{2\sqrt{\mu}}\right)\right| < cexp\left(-\frac{\xi_l^2}{8\mu}\right).$$

Multidimensional and angular parabolic boundary layer functions have an estimate:

$$\left|Y^{r+2,l}(N_{r+2,l})\right| < cexp\left(-\frac{\eta_{r+2}^2 + \eta_l^2}{8\tau_1}\right),$$
$$\left|w^{r+2,l}(M_{r+2,l})\right| < cexp\left(-\frac{\xi_{r+2}^2 + \xi_l^2}{8\mu}\right).$$

The boundary layer functions with rapidly oscillating exponential and power type of change:

$$c(x,t)\exp(\tau_2),\ \sigma = \int_0^t e^{\frac{i[\theta(s)-\theta(0)]}{\varepsilon}}ds.$$

In addition, the asymptotic contains the product of the abovementioned boundary layer functions.

Repeating the above process, we construct a partial sum:

$$\widetilde{u}_{\varepsilon n}(M) = \sum_{i=0}^{n} \varepsilon^{\frac{i}{2}} u_i(M). \tag{36}$$

2.6 Assessment of remainder term

Substituting the function $\widetilde{u}(M,\varepsilon) = u_{\varepsilon n}(M) + \varepsilon^{n+\frac{1}{2}} R_{\varepsilon n}(M)$ into problem (Eq. (24)), then taking into account the iterative tasks of (Eq. (27)) and (Eq. (29)), we obtain the following problem for the remainder term $R_{\varepsilon n}(M)$:

$$\widetilde{L}_\varepsilon R_{\varepsilon n}(M) = g_n(M,\varepsilon),\ R_{\varepsilon n}(M)|_{t=0} = R_{\varepsilon n}(M)|_{x_l=r-1,\ \xi_r=0,\ \eta_k} = 0,\ r = 1,2;\ k = \overline{1,4}, \tag{37}$$

where $g_n(M,\varepsilon) = -i\theta'(t)\partial_{\tau_2}u_{n-1} - \varepsilon^{\frac{1}{2}} i\theta'(t)\partial_{\tau_2}u_n(M) - T_1u_{n-1}(M) - \varepsilon^{\frac{1}{2}}T_1u_n(M) - (D_\sigma - L_\eta)\sum_{k=0}^{3}\varepsilon^{\frac{k}{2}}u_{n-3+k}(M) + L_\eta\sum_{k=0}^{5}\varepsilon^{\frac{k}{2}}u_{n-5+k}(M) + \Delta_a\sum_{k=0}^{7}\varepsilon^{\frac{k}{2}}u_{n-7+k}(M)$.

We put in both parts (Eq. (38)) $\chi = \psi(x,t,\varepsilon)$ considering (Eq. (25)), with respect to

$L_\varepsilon R_{\varepsilon n}(x,t,\varepsilon) = g_{\varepsilon n}(x,t,\varepsilon),\ R_{\varepsilon n}|_{t=0} = 0,\ R_{\varepsilon n}|_{\partial\Omega=0}$.

By virtue of the above constructions, the function is $\left|g_{\varepsilon n}(x,t,\varepsilon)\right| < c,\ \forall(x,t) \in$;

therefore, applying the maximum principle, an estimate is established:

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$$|R_{\varepsilon n}(x,t,\varepsilon)| < c.$$

Thus, we have proven the following:

Theorem 4. Suppose that the conditions (1)–(3) are satisfied. Then, using the above method for solving $u(x,t,\varepsilon)$ of the problem (Eq. (20)), a regularized series (Eq. (26)) such that $\forall n = 0, 1, 2, \ldots$ can be constructed, and for small enough $\varepsilon > 0$, inequality is fair:

$$|u(x,t,\varepsilon) - u_{\varepsilon n}(x,t,\varepsilon)| = |R_{\varepsilon n}\ (x,t,\varepsilon)| < c\varepsilon^{n+\frac{1}{2}},$$

where $\boldsymbol{c}$ is independent of $\boldsymbol{\varepsilon}$.

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